Infinite order Lorenz dominance for fair multiagent optimization

Boris GOLDEN

Polytechnique Scool
France

Boris.Golden@polytechnique.edu

Patrice PERNY

University of Paris 6
France

Patrice.Perny@lip6.fr

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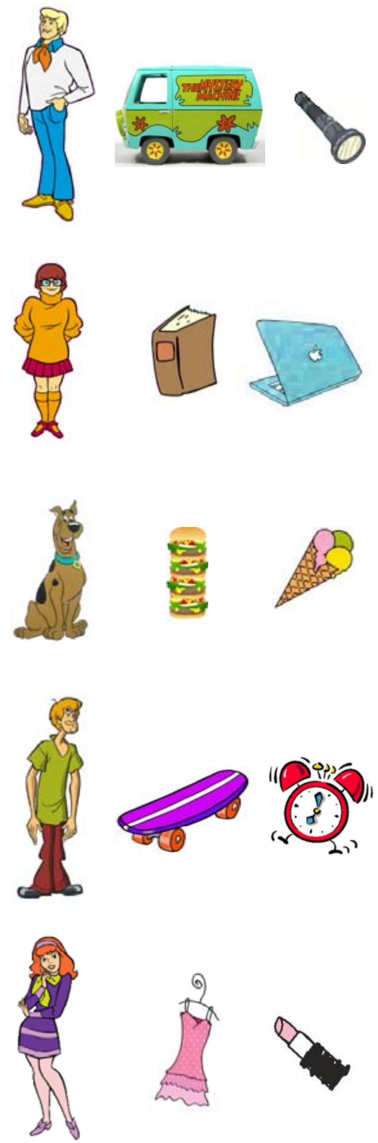
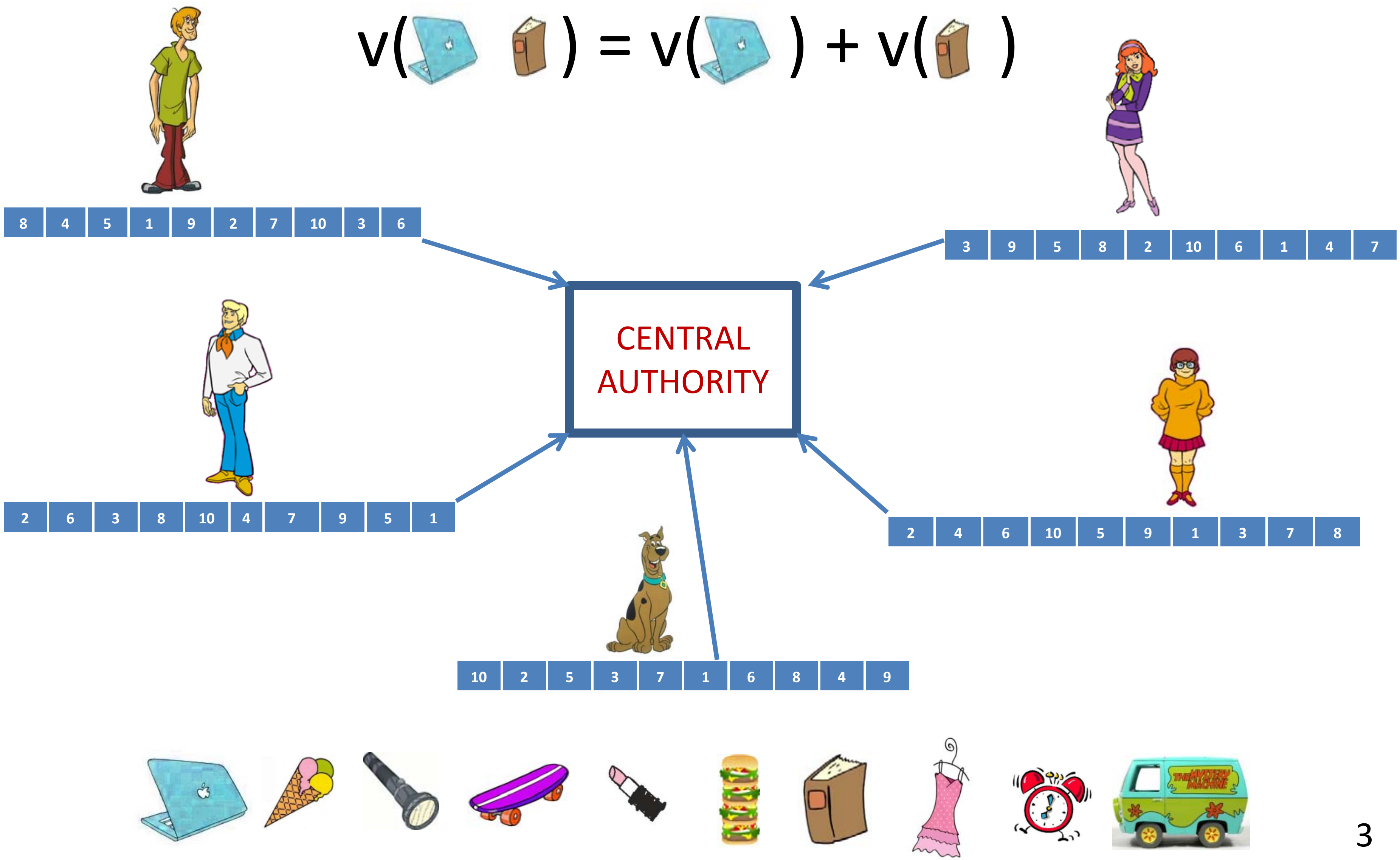
Multiagent Fair Allocation Problems

Fair Allocation

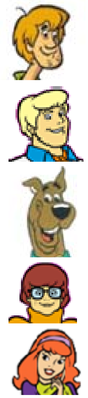
Santa Claus Problem
Resource allocation
Task allocation
Conference paper assignement



Additivity and communication costs



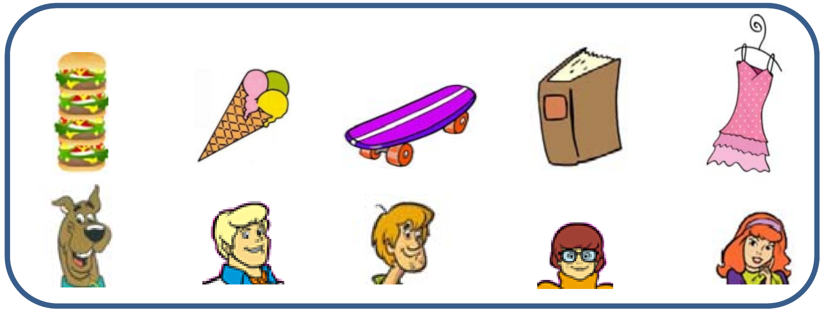
Fairness in assignment problems



5	8	(4)	9	7
1	(3)	2	7	8
(3)	9	2	9	5
10	1	3	(3)	4
5	1	7	7	(3)

1 1 2 3 7

$\Sigma=14$ max = 7

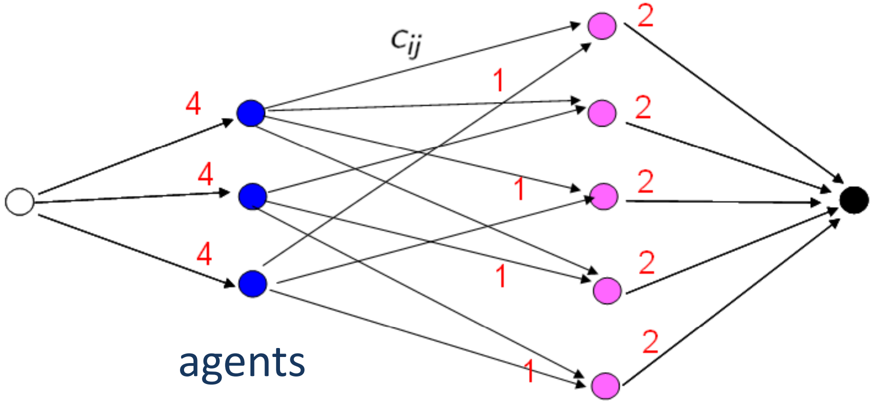


3 3 4 3 3

$\Sigma=16$ max = 4

Modeling assignment problems

Network Flow



Linear Programming

$$\text{Min} \;\; x_i = \sum_{j=1}^{m} c_{ij} z_{ij}, \quad i = 1, \ldots, n$$

$$s.t. \begin{cases} l'_j \leq \sum_{i=1}^{n} z_{ij} \leq u'_j & j = 1, \ldots, m \\ l_i \leq \sum_{j=1}^{m} z_{ij} \leq u_i & i = 1, \ldots, n \\ z_{ij} \in \{0, 1\} & \forall i, \forall j \end{cases}$$

Min Cost Max flow $\rightarrow$ does not guarantee that optimal solutions will be fair!
Minimax optimization $\rightarrow$ focuses on the least satisfied agent (9,9,9,9,9) < (10,1,1,1,1)

Multiobjective optimization $\rightarrow$ too much Pareto-optimal solutions

What else? $\rightarrow$ Fairness in Social Choice

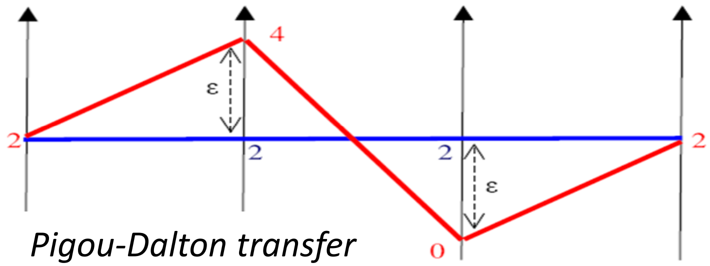
Fairness and Lorenz Dominances

Lorenz vector

For any cost vector $x \in \mathbb{R}^n_+$, we define :
$L(x) = (x_{(1)}, x_{(1)} + x_{(2)}, \ldots, x_{(1)} + x_{(2)} + \ldots + x_{(n)})$
where $x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(n)}$

Generalized Lorenz Dominance [Shorrocks, 83]

$\forall x, y \in \mathbb{R}^n_+,\ x \prec_L y \iff L(x) \prec_P L(y)$



(10, 10, 10) $<_L$ (12, 9, 10) because
(10, 20, 30) $<_P$ (12, 22, 31)

Théorème [Hardy, Littlewood and Polya 29, Chong 76]

For all $x, y \in \mathbb{R}^n_+$, if $x \prec_P y$, or if x obtains from y by a Pigou-Dalton transfer, then $x \prec_L y$. Conversely, if $x \prec_L y$, then there exists a sequence of admissible transfers and/or Pareto-improvements to transform y into x.

(10, 10, 10) $<_L$ (12, 9, 10)
↘ ↗
(11, 9, 10)

$<_L$ refines Pareto dominance

$<_L$ favours well-balanced solutions

Problem

As for Pareto-optimal solutions, the number of Lorenz-optimal solutions grows (in worst case) exponentially with the size of the problem

Refining Lorenz Dominance

I) **Gini Social Evaluation Function [Gini 21, Weymark 81]**

$$G(x) = \frac{1}{n^2}\sum_{i=1}^{n}(2(n-i)+1)x_{(i)} = \frac{2}{n^2}\sum_{i=1}^{n} L_i(x)$$

II) **k-th order Lorenz Dominance : $\prec_L^k$**

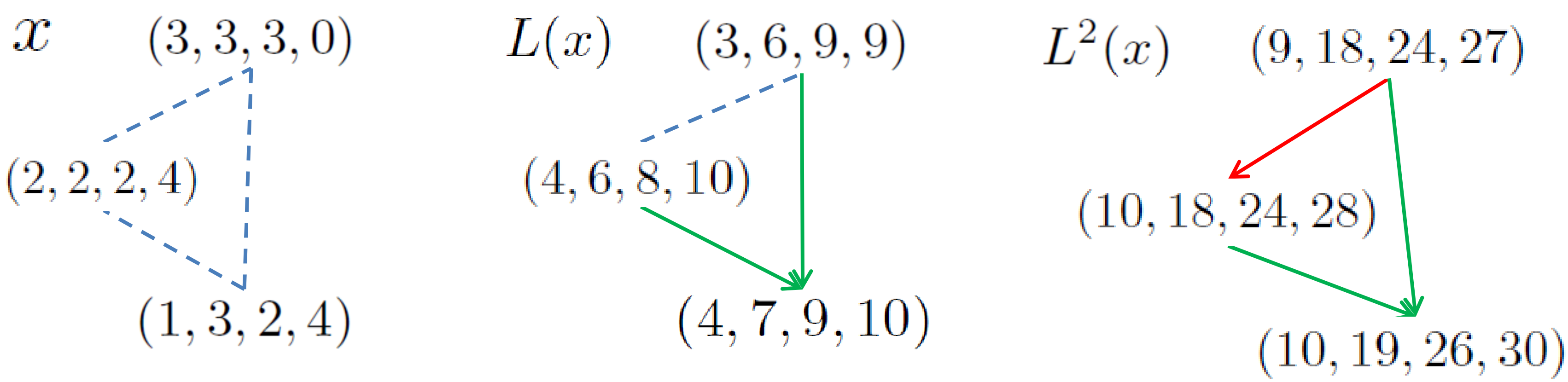
$$L^k(x) = \begin{cases} L(x) & \text{if } k = 1 \\ L(L^{k-1}(x)) & \text{if } k > 1 \end{cases}$$

$$\forall x, y \in \mathbb{R}_+^n,\ x \prec_L^k y \iff L^k(x) \prec_P L^k(y)$$

Infinite order

$$\prec_L^\infty = \bigcup_{k \geq 1} \prec_L^k$$

($\prec_L^k \subseteq \prec_L^{k+1}$, *nested orders*)



L-iteration mechanism

i) Progressive refinements ii) preserves Pigou-Dalton transfers iii) Decisive?

∞-order Lorenz Dominance

Convergence and decisiveness of the L-iteration mechanism?

A representation theorem for L^∞-dominance

$\forall x, y \in \mathbb{R}^n_+,\ x \prec^\infty_L y \iff \mathcal{W}(x) < \mathcal{W}(y)$

where $\mathcal{W}(x) = \sum_{k=1}^{n} \sin\left(\frac{(n+1-k)\pi}{2n+1}\right) x_{(k)}$

W(x) is a Generalized Gini Social Evaluation Function (Weymark, 86)

Ordered average operation with *decreasing weights*

W(*x*) = w' L(x)   with positive coefficients

W(*x*) is Schur convex (Transfer principle)

Fair Assignement problem

$$\text{Min}\quad \mathcal{W}(x) = \sum_{k=1}^{n} \sin\left(\frac{(n+1-k)\pi}{2n+1}\right) x_{(k)}$$

$$s.t.\ \begin{cases} x_i = \sum_{j=1}^{m} c_{ij} z_{ij} & i = 1, \ldots, n \\ l'_j \leq \sum_{i=1}^{n} z_{ij} \leq u'_j & j = 1, \ldots, m \\ l_i \leq \sum_{j=1}^{m} z_{ij} \leq u_i & i = 1, \ldots, n \\ z_{ij} \in \{0, 1\} & \forall i, \forall j \end{cases}$$

NP-Hard

Linear reformulation as a MIP

A MIP formulation for fair optimization

$$y = (y_1, \ldots, y_n) \qquad y_{(1)} \geq y_{(2)} \cdots \geq y_{(n)}$$

$$\mathrm{OWA}(y) = \sum_{k=1}^{n} w_i y_{(k)} = \sum_{k=1}^{n} w'_k L_k(y) \qquad w' = (w_1 - w_2, \ldots, w_{n-1} - w_n, w_n) > 0$$

$L_k(y) =$

$$\begin{array}{ll} \text{Max } \sum_{i=1}^{n} \alpha_i^k y_i & \\ \sum_{i=1}^{n} \alpha_i^k = k & \\ 0 \leq \alpha_i^k \leq 1 & i = 1 \ldots n \end{array}$$

$$\begin{array}{ll} \text{Min } k r_k + \sum_{i=1}^{n} b_i^k & \\ r_k + b_i^k \geq y_i & \\ b_i^k \geq 0 & i = 1 \ldots n \end{array}$$

dual

$$\text{Min} \sum_{k=1}^{p} w'_k \left(k r_k + \sum_{i=1}^{n} b_i^k \right)$$

$$r_k + b_i^k \geq y_i$$

$$b_i^k \geq 0$$

Final MIP Formulation

$$\text{Min} \quad \sum_{k=1}^{n} w'_k \left(k \times r_k + \sum_{i=1}^{n} b_{ik} \right)$$

$$s.t. \begin{cases} l'_j \leq \sum_{i=1}^{n} z_{ij} \leq u'_j & j = 1, \ldots, m \\ l_i \leq \sum_{j=1}^{m} z_{ij} \leq u_i & i = 1, \ldots, n \\ r_k + b_{ik} \geq \sum_{j=1}^{m} c_{ij} z_{ij} & \forall i, k = 1, \ldots, n \\ b_{ik} \geq 0 & \forall i, k \\ z_{ij} \in \{0, 1\} & \forall i, \forall j \end{cases}$$

Numerical Tests
with CPLEX 11.1(times in s)

Assignment 1
n= m
Scale = [1..1000]
li=ui=l'i=u'i=1

m	t
10	.01
20	.09
30	.33
40	1.52
50	5.14
60	16.1
70	34.0
80	81.8
90	136
100	275

Assignment 2
n= m
Scale = [1..20]
li=ui=l'i=u'i=1

m	t
100	.93
200	3.65
300	17.4
400	52.8
500	104
600	161
700	390
800	482
900	843
1000	>1000

Paper assignment
n= m/4
Scale = [1, 5]
li=0, ui=9 l'i=u'i=2

m	t
200	3.51
300	5.63
400	13.9
500	35.7
600	79.4
700	148
800	303
900	478
1000	904
1100	>1000

n : number of agents m : number of items