Infinite order Lorenz dominance for fair multiagent optimization

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ABSTRACT

This paper deals with fair assignment problems in decision contexts involving multiple agents. In such problems, each agent has its own evaluation of costs and we want to find a fair compromise solution between individual point of views. Lorenz dominance is a standard decision model used in Economics to refine Pareto dominance while favoring solutions that fairly share happiness among agents. In order to enhance the discrimination possibilities offered by Lorenz dominance, we introduce here a new model called *infinite order Lorenz dominance*. We establish a representation result for this model using an ordered weighted average with decreasing weights. Hence we exhibit some properties of infinite order Lorenz dominance that explain how fairness is achieved in the aggregation of individual preferences. Then we explain how to solve fair assignment problems of m items to n agents, using infinite order Lorenz dominance and other models used for measuring inequalities. We show that this problem can be reformulated as a 0-1 non-linear optimization problems that can be solved, after a linearization step, by standard LP solvers. We provide numerical results showing the efficiency of the proposed approach on various instances of the paper assignment problem.

Categories and Subject Descriptors

J.4 [**Social and Behavioral Sciences**]: Economics; I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—*Multiagent Systems*; G.1.5 [**Numerical Analysis**]: Optimization—*integer programming, linear programming*

General Terms

Algorithms, Economics

Keywords

Multiagent optimization, Fairness

1. INTRODUCTION

Fairness of decision procedures is often considered as an important issue in decision problems involving multiple agents. Although not always formalized precisely, this normative principle generally refers to the idea of favoring solutions that fairly share happiness or dissatisfaction among agents. More formally, when comparing two cost vectors x and y (one component by agent), claiming that "x is more fair than y" bears to the vague notion that the components of x are "less spread out" or "more nearly equal" than the components of y are. This intuitive notion leaves room for many different definitions. The field has been explored by mathematicians who developed a formal theory of majorization [10] and by economists who studied the axiomatic foundations of inequality measures (for a synthesis see [11, 15]).

This body of knowledge has now a significant impact in computer sciences where many optimization problems require to incorporate the idea of fairness or equity in the definition of objectives. Let us mention for example multiagent job-shop scheduling problems, knapsack sharing problems, equitable approaches to location problems [12], fair bandwidth assignment, or any other resource allocation problem. This is also the case in the field of Artificial Intelligence where the notions of fairness and envy-freeness appear in various multiagent problems such as fair division of indivisible goods and combinatorial auctions [2, 3], paper assignment problems [7], marriage problems in social networks [6].

Example 1. *Let us consider a simple fair division problem where 5 items must be assigned to 5 agents. Every item is assigned to exactly one agent and each agent is assigned exactly one item. We want to find an assignment that fairly shares costs between agents, the costs being given by the following matrix of general term c_{ij} representing the cost of assigning item j to agent i:*

$$C = \begin{pmatrix} 5 & 8 & (4) & 9 & \mathbf{7} \\ \mathbf{1} & (3) & 2 & 7 & 8 \\ (3) & 9 & \mathbf{2} & 9 & 5 \\ 10 & 1 & 3 & (\mathbf{3}) & 4 \\ 5 & \mathbf{1} & 7 & 7 & (3) \end{pmatrix}$$

Any solution to this problem is a permutation that can be characterized by a square matrix Z of size 5 containing boolean variables z_{ij} where $z_{ij} = 1$ if and only if item j is assigned to agent i, Z having exactly one 1 in each row and column. To solve this multiagent assignment problem using standard optimization techniques, we could be interested in minimizing the average level of dissatisfaction among individuals, or equivalently the sum of individual dissatisfactions where the dissatisfaction of agent i is defined by $x_i = \sum_j c_{ij} z_{ij}$. This amounts to minimizing the linear function $\sum_i \sum_j c_{ij} z_{ij}$, a classical matching problem which can be solved in polytime with the Hungarian method. Here

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the optimal solution is given by setting to 1 variables z_{ij} corresponding to costs c_{ij} in bold in the C matrix. The associated dissatisfaction vector is given by $(7, 1, 2, 3, 1)$ which yields 14 as overall cost. However, this solution does not seem very fair. Although the average cost is below 3, one agent receives 7 whereas another gets 1.

If we consider now another permutation given by numbers into brackets in the cost matrix, we get a much preferable dissatisfaction profile regarding equity. For a slightly higher overall cost (16), we indeed obtain a significantly better balanced dissatisfaction profile: $(4, 3, 3, 3, 3)$. This solution actually minimizes the dissatisfaction of the least satisfied agent (min-max criterion) and the solution is here fully satisfactory. However, focusing on the least satisfied agent is not always convenient. It provides a pessimistic view on agents' satisfactions; moreover it is not very discriminating since multiple solutions remain equivalent from a worst case analysis point of view, even if they offer different perspectives to all but the least satisfy agent. The worst case can even mask very different situations as shown by this second example:

Example 2. *We consider an assignment problem with the following cost matrix:*

$$C' = \begin{pmatrix} 9 & 10 & (9) & 9 & \mathbf{10} \\ \mathbf{1} & (4) & 2 & 7 & 8 \\ (4) & 9 & \mathbf{2} & 9 & 5 \\ 10 & 1 & 3 & (\mathbf{2}) & 4 \\ 5 & \mathbf{1} & 7 & 7 & (4) \end{pmatrix}$$

Here, the optimal solution obtained with respect to the min-max criterion is given by numbers into brackets in the matrix. The associated dissatisfaction vector is $(9, 4, 4, 2, 4)$. However, in this case, the min-max solution might not be the best one. We could prefer sacrificing the least satisfied agent (who is apparently difficult to satisfy) so as to get better costs for the other agents. Hence vector $(10, 1, 2, 2, 1)$ that derives from positions in bold in matrix C' should be preferred to the previous one.

These examples show that simple objectives like min-sum or min-max are not perfectly suited to fair optimization problems. In this paper we propose a more sophisticated model that attaches more importance to least satisfied agents without forgetting the other agents. It is based on an extension of a partial dominance concept known as *Lorenz Dominance* in Social Choice Theory and used for the measurement of inequalities. Our aim in this paper is to introduce this model and its main properties, and then elaborate a computationally efficient procedure using this model to generate fair solutions in multiagent assignment problems. For application purpose, we will consider one to one assignment problems as in Examples 1 and 2, but also many to many assignment problems such as conference paper assignment problems. The multiagent problems discussed in this paper concern the case of centralized information. We assume that a central authority is responsible of computations and assignment of items. This is the case in various auctions problems and in conference paper assignment problems. It would also be interesting to study similar problems in decentralized contexts where the final assignment emerges from a sequence of local decisions of uncoordinate agents having only a partial view on the problem [3, 8]. Such problems are beyond the scope of this paper.

The paper is organized as follows: in Section 2 we recall some basic concepts used in Social Choice theory for the measurement of inequalities. In order to minimize agents' dissatisfaction while preserving fairness in assignment, we introduce the notion of infinite order Lorenz dominance as a refinement of Pareto and Lorenz dominance concepts. Then we establish a representation result for infinite order Lorenz dominance in Section 3, and we present some axiomatic properties of this model. The use of this model in multiagent assignment problems in presented in Section 4. In particular we formulate such problems as non-linear 0-1 optimization problems, we study the problem complexity and present an approach to solve it using mixed integer linear programming. Finally numerical results showing the efficiency of our approach on randomly generated instances are presented, including a model of the paper assignment problem solved for realistic sizes.

2. INEQUALITY MEASUREMENT WITH LORENZ DOMINANCE RELATIONS

2.1 Notations and Definitions

Considering a finite set of agents $N = \{1, \ldots, n\}$, any solution of a multiagent combinatorial problem can be characterized by a cost vector $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n_+$ whose i^{th} component represents the cost of solution x with respect to agent i. Hence, the comparison of solutions reduces to the comparison of their cost vectors. In this framework, the following definitions are useful:

Definition 1. *The Weak-Pareto dominance relation on cost vectors of $\mathbb{R}^n_+$ is defined, for all $x, y \in \mathbb{R}^n_+$ by:*

$$x \precsim_P y \iff [\forall i \in N, x_i \leq y_i)]$$

The Pareto dominance relation (P-dominance for short) on cost vectors of $\mathbb{R}^n_+$ is defined as the asymmetric part of $\precsim_P$:

$$x \prec_P y \iff [x \precsim_P y \text{ and } \text{not}(y \precsim_P x)]$$

Remark that $x \prec_P y$ means that x is preferred to y (x is less costly than y) since x and y are cost vectors representing individuals' dissatisfactions. Within a set X we say that x is *P-dominated* when $y \prec_P x$ for some y in X, and *P-non-dominated* when there is no y in X such that $y \prec_P x$.

In order to decide whether a solution is better than another, we have to define a transitive preference relation $\precsim$ on cost vectors such that $x \precsim y$ when cost vector x is preferred to cost vector y. Let us introduce now the minimal requirements that such a relation $\precsim$ should satisfy to be seen as a reasonable synthesis of agents' opinions, favoring both efficiency and equity in comparisons. Firstly, we assume that all agents have the same importance. Hence, the following axiom formalizes the fact that all agents are treated equivalently:

Symmetry. For all $x \in \mathbb{R}^n_+$, for any permutation π of $\{1, \ldots, n\}$, $(x_{\pi(1)}, \ldots, x_{\pi(n)}) \sim (x_1, \ldots, x_n)$, where $\sim$ is the indifference relation defined as the symmetric part of $\precsim$.

In relation $\precsim$ we both want to capture the ideas of fairness and efficiency in cost-minimization. For this reason, $\precsim$ is expected to satisfy the following axioms:

P-Monotonicity. For all $x, y \in \mathbb{R}^n_+$, $x \precsim_P y \Rightarrow x \precsim y$ and $x \prec_P y \Rightarrow x \prec y$, where $\succ$ is the strict preference relation defined as the asymmetric part of $\precsim$. P-monotonicity is a natural unanimity principle enforcing consistency with P-dominance.

Now the idea of fairness in comparisons is based on the following transfer principle:

Transfer Principle. Let $x \in \mathbb{R}_+^n$ such that $x_i > x_j$ for some i, j. Then for all ε such that $0 < \varepsilon < x_i - x_j$, $x - \varepsilon e_i + \varepsilon e_j \prec x$ where e_i (resp. e_j) is the vector whose i^{th} (resp. j^{th}) component equals 1, all others being null.

This axiom captures the idea of fairness as follows: if $x_i > x_j$ for some cost vector $x \in \mathbb{R}_+^n$, slightly improving (here decreasing) component x_i to the detriment of x_j while preserving the mean of the costs would produce a better distribution of costs and consequently improve the overall cost of the solution for the collection of agents. For example if $y = (9, 10, 9, 10)$ and $x = (11, 10, 7, 10)$ then the transfer principle implies $y \prec x$. Vector y is preferred because there exists a transfer of size $\epsilon = 2$ to pass from x to y. Note that using a similar transfer of size greater than 11 - 7 = 4 would increase inequality in terms of costs. This explains why the transfers must have a size $\varepsilon < x_i - x_j$. Such transfers are said to be *admissible* in the sequel. They are known as *Pigou-Dalton transfers* in Social Choice Theory, where they are used to reduce inequality of income distributions over a population (see [15] for a survey).

Note that the transfer principle possibly provides arguments to discriminate between vectors having the same average cost but does not apply in the comparison of vectors having different average costs. Hopefully, the possibility of discriminating is improved when combining the Transfer Principle with P-monotonocity. For example, to compare $w = (8, 10, 9, 10)$ and $z = (11, 10, 7, 12)$ we can use vectors x and y introduced above and observe that $w \prec y$ (P-Monotonicity), $y \prec x$ (Tranfer Principle explained above) and $x \prec z$ (P-Monotonicity). Hence $w \prec z$ by transitivity. In order to better characterize those vectors that can be compared using combinations of P-monotonicity and Transfer Principle we recall the definition of *Generalized* Lorenz vector and related concepts :

Definition 2. *For all $x \in \mathbb{R}_+^n$, the* Generalized Lorenz Vector *associated to x is the vector:*

$$L(x) = (x_{(1)}, x_{(1)} + x_{(2)}, \ldots, x_{(1)} + x_{(2)} + \ldots + x_{(n)})$$

where $x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(n)}$ represents the components of x sorted by decreasing order. The j^{th} component of $L(x)$ is $L_j(x) = \sum_{i=1}^{j} x_{(i)}$.

Definition 3. *The Generalized Lorenz dominance relation (L-dominance for short) on $\mathbb{R}_+^n$ is defined by:*

$$\forall x, y \in \mathbb{R}_+^n,\ x \precsim_L y \iff L(x) \precsim_P L(y)$$

Within a set X, element x is said to be *L-dominated* when $y \prec_L x$ for some y in X, and *L-non-dominated* when there is no y in X such that $y \prec_L x$.

The notion of Lorenz dominance was initially introduced to compare vectors with the same average cost and its link to the transfer principle was established by Hardy, LittleHood and Polya [10]. The generalized version of L-dominance considered here is a classical extension allowing vectors with different averages to be compared (see [16]). In order to establish the link between Generalized Lorenz dominance and preferences satisfying combination of P-Monotonocity, Symmetry and Transfer Principle we recall a result of Chong [4] (see also [10] and [16]):

Theorem 1. *For any pair of distinct vectors $x, y \in \mathbb{R}_+^n$, if $x \prec_P y$, or if x obtains from y by a Pigou-Dalton transfer, then $x \prec_L y$. Conversely, if $x \prec_L y$, then there exists a sequence of admissible transfers and/or Pareto-improvements to transform y into x.*

For example we have: $L(w) = (10, 20, 29, 37) \prec_P L(z) = (12, 23, 33, 40)$ which directly proves the existence of a sequence of Pareto improvements and/or admissible transfers passing from z to w. This theorem establishes L-dominance as the minimal transitive relation (with respect to set inclusion) satisfying simultaneously P-Monotonicity, Symmetry and the Transfer Principle. Hence, the subset of L-non-dominated elements defines the best candidates to optimality in fair optimization problems.

Due to P-monotonicity, the set of L-non-dominated elements is included in the set of Pareto optimal vectors. Unfortunately, in multi-objective combinatorial optimization problems, the set of L-non-dominated solutions can be huge (see [14]). This problems occurs also in multiagent assignment problems. As we will see later in Example 5, there exists family of instances where the number of L-non-dominated cost vectors grows exponentially with the size of the problem. This is the reason why we introduce in the next section more discriminating dominance concepts that extend L-dominance to richer preference structures.

Other attempts in this direction have been proposed in Social Choice Theory. The most common way is resorting to a Schur-convex function ψ to construct a weak-order defined by $x \precsim y \Leftrightarrow \psi(x) \leq \psi(y)$. A Schur-convex function (also known as order-preserving function) is a function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\forall x, y \in \mathbb{R}^n$, $x \precsim_L y \implies \psi(x) \leq \psi(y)$. For example, every function that is convex and symmetric is also Schur-convex. Well known examples of such functions are S-Gini indices and more generally instances of Yaari's model [18] of the following form:

Example 3. *The Yaari's Social Welfare Functions of the following form are Schur-convex:*

$$W_f(x) = \sum_{i=1}^{n} \left[f\left(\frac{n-i+1}{n}\right) - f\left(\frac{n-i}{n}\right) \right] x_{(i)} \quad (1)$$

where f is a strictly increasing continuous function such that $f(0) = 0$ and $f(1) = 1$. S-Gini indices are particular instances obtained for $f(z) = z^\delta$, $\delta > 1$, see [5].

There are other ways of refining Lorenz dominance. We can import some ideas from the literature on Decision Making under risk, where people are interested in comparing probability distributions in terms of risk. In this context, the counterpart of Lorenz dominance is the second-order stochastic dominance (SSD for short) that defines a partial order on probability distributions. The SSD model does not permit to compare any pair of distributions, but it can be refined by stochastic dominances of higher orders, each of them refining the previous one. The ultimate result of this process is named infinite order stochastic dominance (see [9]). The next subsection proposes the construction of progressive refinements of L-dominance using similar mechanisms.

2.2 Infinite Order Lorenz Dominance

Refinement of Lorenz dominance can be obtained by iterating $L(.)$ transformation so as to define higher order L-dominance relations. Observing indeed that P-monotonicity holds for L-dominance (see Theorem 1), L-dominance appears as a refinement of Pareto dominance. Whenever x and y cannot be compared in terms of P-dominance we compare instead $L(x)$ and $L(y)$. If no Pareto dominance holds, the indetermination might be solved by comparing $L^2(x) = L(L(x))$ and $L^2(y) = L(L(y))$. This process can be iterated mechanically to higher levels with the aim of reducing incomparability. This leads to consider k^{th} order Lorenz vector $L^k(x)$ defined by:

$$L^k(x) = \begin{cases} x & \text{if } k = 0 \\ L(L^{k-1}(x)) & \text{if } k > 1 \end{cases}$$

and the k^{th} order Lorenz dominance defined by:

$$\forall x, y \in \mathbb{R}^n_+,\ x \precsim^k_L y \iff L^k(x) \precsim_P L^k(y)$$

Then we define strict *infinite order dominance* (strict L^∞-dominance for short) as follows[1]:

$$\prec^\infty_L = \bigcup_{k \geq 1} \prec^k_L$$

Note that, $\prec^0_L$ and $\prec^1_L$ correspond to P-dominance and L-dominance respectively. Then, due to P-monotonicity, $x \prec^k_L y \Rightarrow x \prec^{k+1}_L y$ for any k and relations $\prec^k_L$ form a *nested* sequence of strict partial orders. This suggests that $\prec^\infty_L$ might be computed, for any pair $x, y \in \mathbb{R}^n_+$ by Algorithm 1 given below:

Algorithm 1: Testing strict L^∞-dominance

```
u ← x;
v ← y;
while [not(u ≺_P v or v ≺_P u)] do
    u ← L(u);
    v ← L(v);
end
if (u ≺_P v) then x ≺_L^∞ y;
if (v ≺_P u) then y ≺_L^∞ x
```

For example, consider a 4 agents problem with 3 Pareto optimal feasible vectors $x = (3, 2, 3, 2), y = (3, 3, 3, 0)$ and $z = (1, 3, 2, 4)$. We have $L(x) = (3, 6, 8, 10), L(y) = (3, 6, 9, 9)$ and $L(z) = (4, 7, 9, 10)$. Hence we get $x \prec^\infty_L z$ and $y \prec^\infty_L z$. We need to go one step ahead to compare x and y. We get $L^2(x) = (10, 18, 24, 27)$ and $L^2(y) = (9, 18, 24, 27)$, therefore $y \prec^\infty_L x$.

Note that our definition of infinite order Lorenz dominance assumes that the vectors to be compared are cost vectors. It does not fit for utility vectors. A simple way of adapting our approach to compare two utility vectors $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ according to infinite order dominance is to check whether $(M - u_1, \ldots, M - u_n) \prec^\infty_L (M - v_1, \ldots, M - v_n)$ for an arbitrary M chosen greater than all u_i and v_i, $i = 1, \ldots, n$. This adaptation is consistent with the definition of $\prec^\infty_L$ for cost vectors and does not depend on the choice of M.

Algorithm 1 tries to discriminate between vectors that were not discriminated by Lorenz dominance. However, nothing proves that the algorithm terminates for all pairs of vectors. Moreover the mechanical iteration of Lorenz dominance used to introduce the model is not easy to manipulate when we study the properties of the model. We need another characterization of $\prec^\infty_L$ to be able to propose a fully operational decision procedure and to be able to better understand the role of each agent in the decision process. For this reason, in the following section, we characterize the vectors that can be discriminated by Algorithm 1. We also provide a direct mathematical definition of strict L^∞-dominance making it possible to compare any pair of vectors in $O(n \log(n))$.

3. SOME PROPERTIES OF INFINITE ORDER LORENZ DOMINANCE

3.1 A Representation Theorem

In this section, we establish a representation result for strict L^∞-dominance. We present an algebraic reformulation of Lorenz vectors and establish technical lemmas; the main result will follow immediately.

For $x \in \mathbb{R}^n_+$, we define $x^\uparrow$ as the vector resulting from sorting the components of x in increasing order, that is: $x^\uparrow = (x_{(n+1-i)})_{i=1\ldots n}$ since we have defined $(.)$ as the permutation sorting the components of x in decreasing order. As the definition of $L(.)$ respects the Symmetry Axiom, we have: $L(x^\uparrow) = L(x)$. We now introduce the $n \times n$ matrix:

$$\mathcal{L} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & \vdots \\ 0 & \iddots & & 1 \\ 1 & \cdots & \cdots & 1 \end{pmatrix}$$

defined by: $l_{ij} = 1$ if $i + j > n$, 0 otherwise.

Proposition 1. *For* $x \in \mathbb{R}^n_+$, $\forall k, L^k(x) = \mathcal{L}^k.x^\uparrow$

Schetch of the proof. For a vector y whose components are sorted in increasing order, it is immediate to verify that $L(y)$ (the Lorenz vector of y) is equal to $\mathcal{L}.y$ (product of the matrix $\mathcal{L}$ and the vector y). Therefore, for any vector x, we have: $L(x) = L(x^\uparrow) = \mathcal{L}.x^\uparrow$. As $L(x)$ is a vector whose components are sorted in increasing order, the equality holds at any order: $\forall k, L^k(x) = \mathcal{L}^k.x^\uparrow$ □

$\mathcal{L}$ being a symmetric real matrix, the finite-dimensional spectral theorem applies and we can find P, an orthogonal matrix (such that ${}^tP = P^{-1}$) and n real eigenvalues $\lambda_1 \ldots \lambda_n$ (duplicated according to their multiplicity, with $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$) such that: $\mathcal{L} = {}^tP\ Diag(\lambda_1 \ldots \lambda_n)\ P$ and therefore $\mathcal{L}^k = {}^tP\ Diag(\lambda^k_1 \ldots \lambda^k_n)\ P$, $Diag(a_1 \ldots a_n)$ being the diagonal matrix of elements $a_1, a_2, \ldots, a_n$.

Let $w = \frac{\pi}{2n+1}$:

Lemma 1. *The* n *eigenvalues of* $\mathcal{L}$ *are* $\lambda_k = \frac{(-1)^{k+1}}{2\sin\left(\frac{(2k-1)w}{2}\right)}$ *for* $k \in [1; n]$*, with eigenvectors* $V_k = \Big(\sin\big(i(2k-1)w\big)\Big)_{i=1\ldots n}$

Schetch of the proof. We introduce $A = 2\ \mathrm{Id} - \mathcal{L}^{-2}$, which is classical to diagonalize ($\mathcal{L}^{-2}$ is the square of the inverse matrix of $\mathcal{L}$). The n eigenvalues of A are $2\cos((2k-1)w)$, $k =$

[1]Although L-dominance can be seen as a particular instance of second order stochastic dominance (assuming a uniform probability distribution on agents), the notion of infinite order Lorenz dominance we introduce here must not be confused with infinite order stochastic dominance that results from a different construction.

$1 \dots n$, with eigenvectors $\left(\sin\left(i(2k-1)w\right)\right)_{i=1\dots n}$. These n eigenvalues of A are in $(0,2)$ and have different absolute values. Using the relation between A and $\mathcal{L}$, we are able to determine the eigenvalues of $\mathcal{L}$ and its eigenvectors. □

Decomposing $\mathcal{L}^k$ into n lines $\mathcal{L}_1^k \dots \mathcal{L}_n^k$, we have:

$$x \precsim_L^k y \quad \Leftrightarrow \quad \mathcal{L}^k.x^\uparrow \precsim_P \mathcal{L}^k.y^\uparrow \quad \Leftrightarrow \quad \begin{cases} \mathcal{L}_1^k.x^\uparrow \leq \mathcal{L}_1^k.y^\uparrow \\ \vdots \\ \mathcal{L}_n^k.x^\uparrow \leq \mathcal{L}_n^k.y^\uparrow \end{cases}$$

Thus, the k^{th} order Lorenz dominance can be rewritten as the intersection of n orders. To determinate $\prec_L^\infty$, we will express these n orders and prove that they are equivalent when $k \to \infty$, and that this equivalent admits a limit when $k \to \infty$, limit that can be rewritten as the strict order induced by an OWA function.

Definition 4. *OWA means* ordered weighted average. *It is a family of aggregators introduced by Yager [19] characterized by $W(x) = \sum_{k=1}^n w_k x_{(k)}$. W is a symmetric function of its arguments. The weights w_k do not represent the importance of agents but the attention we pay to agents depending on their rank in the satisfaction order.*

Let E_i be the square matrix of dimension n with all values to 0, excepted a 1 in position (i,i). Let P_i be the i^{th} line of matrix P and $A_i = {}^tP_iP_i$. We have:

$$\mathcal{L}^k = {}^tP\left(\sum_{i=1}^n \lambda_i^k E_i\right)P = \sum_{i=1}^n \lambda_i^k({}^tP_iP_i) = \sum_{i=1}^n \lambda_i^k A_i$$

Lemma 2. *For $x \in \mathbb{R}_+^n$, $L^k(x) \sim \lambda_1^k\ {}^tP_1 \times \mathcal{W}(x)$ where $\mathcal{W}$ is the OWA function whose weights are the components of P_1 in reverse order and $\sim$ is the relation of asymptotical equivalence.*

Proof. By Lemma 1 giving the eigenvalues of $\mathcal{L}$, we have $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$. Moreover, the eigenvector associated to the greatest eigenvalue λ_1 being $V_1 = {}^t P_1 = \left(\sin(iw)\right)_{i=1\dots n} > 0$, we have ${}^tP_1P_1 > 0$ and therefore, when $k \to +\infty$, we have (since for $1 < i \leq n$, $\lambda_i^k = o(\lambda_1^k)$ when $k \to +\infty$) : $\mathcal{L}^k \sim \lambda_1^k\ {}^tP_1P_1$ so that we can write: $\mathcal{L}^k(x) \sim \lambda_1^k\ {}^tP_1(P_1.x^\uparrow)$. $P_1.x^\uparrow$ is a weighted average of the components of the increasing vector $x^\uparrow$. It can be rewritten as an OWA on x, since if we define $\mathcal{W}$ as the OWA criterion whose weights are the components of P_1 in reverse order, we have: $\mathcal{W}(x) = \sum_{i=1}^n (P_1)_{n+1-i}\ x_{(i)} = P_1.x^\uparrow$. We finally obtain the following equivalent of the iteration of Lorenz dominance when $k \to \infty$: $L^k(x) \sim \lambda_1^k\ {}^tP_1 \times \mathcal{W}(x)$ □

Lemma 3. *For $x, y \in \mathbb{R}_+^n$, $\mathcal{W}(x) < \mathcal{W}(y) \Rightarrow x \prec_L^\infty y$.*

Proof. If $\mathcal{W}(x) < \mathcal{W}(y)$, then, as $L^k(x) \sim \lambda_1^k\ {}^tP_1 \times \mathcal{W}(x)$ (by Lemma 2), we have for k suffcently large: $L^k(x) \succ_P L^k(y)$, so $x \succ_L^k y$, and therefore $x \prec_L^\infty y$. □

Lemma 4. *For $x, y \in \mathbb{R}_+^n$, if $\mathcal{W}(x) = \mathcal{W}(y)$, then x and y are incomparable by $\prec_L^\infty$.*

Proof. The equivalent of $L^k(x)$ of Lemma 2 cannot discriminate between two vectors x and y when $\mathcal{W}(x) = \mathcal{W}(y)$. We need to use the other eigenvalues of $\mathcal{L}$ to try to discriminate between these two vectors. As $\mathcal{W}(x) = \mathcal{W}(y)$, we can write, according to the result of Proposition 1:

$$L^k(y) - L^k(x) = \sum_{i=2}^n \lambda_i^k\ {}^tP_iP_i(y^\uparrow - x^\uparrow)$$

If for all j, $P_j(y^\uparrow - x^\uparrow) = 0$, then $P(y^\uparrow - x^\uparrow) = 0$ and $x^\uparrow = y^\uparrow$ (P being an invertible matrix), and therefore there is no strict $\prec_L^\infty$-dominance between x and y, since $L(x) = L(y)$, and they are incomparable by $\prec_L^\infty$.

Else, let j be the first index such that $P_j(y^\uparrow - x^\uparrow) \neq 0$. As ${}^tPP = Id$, for $i > 1$, ${}^tP_1P_i = 0$. But since $P_i \neq 0$ (P being an invertible matrix) and $P_1 > 0$, P_i must have a component i_1 strictly positive and another i_2 strictly negative to verify ${}^tP_1P_i = 0$. But we have, for any component i:

$$L_i^k(y) - L_i^k(x) \sim \lambda_j^k\ {}^t(P_j)_i(P_j(y^\uparrow - x^\uparrow))$$

so for k sufficiently large, the component i_1 of $L^k(y) - L^k(x)$ is strictly positive and i_2 stricly negative; thus $\succ_L^k$-dominance cannot hold and x and y are incomparable by $\prec_L^\infty$. □

We are now in position to formulate our main result:

Theorem 2. *The strict L^∞-dominance has a direct numerical representation using the following ordered weighted average:*

$$\mathcal{W}(x) = \sum_{k=1}^n \sin\left(\frac{(n+1-k)\pi}{2n+1}\right) x_{(k)}$$

This representation is given by the following property:

$$\forall x, y \in \mathbb{R}_+^n,\ x \prec_L^\infty y \iff \mathcal{W}(x) < \mathcal{W}(y)$$

Proof. $\Rightarrow$: if $x \prec_L^\infty y$, then x and y are not incomparable by $\prec_L^\infty$. Therefore, the contrapositive of Lemma 4 insures that $\mathcal{W}(x) \neq \mathcal{W}(y)$. But then, as $x \prec_L^\infty y$, we cannot have $\mathcal{W}(x) > \mathcal{W}(y)$ (since Lemma 3 would apply and imply that $y \prec_L^\infty x$). We finally must have $\mathcal{W}(x) < \mathcal{W}(y)$.

$\Leftarrow$: the proof is straightforward by Lemma 3. □

Actually, to compare two vectors x and y according to strict L^∞-dominance, we do not need to run Algorithm 1. We compute instead $\mathcal{W}(x)$ and $\mathcal{W}(y)$ in $O(n\log(n))$. If $\mathcal{W}(x) \neq \mathcal{W}(y)$, the vector having the smallest score by $\mathcal{W}$ strictly L^∞-dominates the other. Hence, Algorithm 1 would stop after a sufficiently large number of iterations. Whenever $\mathcal{W}(x) = \mathcal{W}(y)$, no strict dominance holds at any order of the iteration of Lorenz dominance. This shows that Algorithm 1 would never terminate in this case. This illustrates the meaning and utility of our representation result.

The last remark deals with vectors incomparable by strict L^∞-dominance: the relation "is incomparable with" is a relation of equivalence (since it means having the same value by function $\mathcal{W}$). Therefore, it is natural to extend the strict L^∞-dominance to a weak order as follows:

Definition 5. *The L^∞-dominance is a weak order extending strict L^∞-dominance as follows:*

$$\forall x, y \in \mathbb{R}_+^n,\ x \precsim_L^\infty y \iff \mathcal{W}(x) \leq \mathcal{W}(y)$$

This numerical representation of L^∞-dominance is very helpful to analyze the axiomatic properties of the model. The fact that $\mathcal{W}(x)$ is an ordered weighted average (OWA) with positive and *strictly decreasing* weights w_k as k increases makes sense. It means that all agents play a role in the evaluation of solutions but, when evaluating a given solution, we attach more importance to least satisfied agents. This is in accordance with the intuitive idea of fairness presented in the introduction. In particular we have $(4,3,3,3,3) \prec_L^\infty (7,1,2,3,1)$ and $(10,1,2,2,1) \prec_L^\infty (9,4,4,2,4)$ as desired in Examples 1 and 2, which outperforms the possibilities of min-sum and min-max criteria.

3.2 Main Properties of L^∞-dominance

We first exhibit two important propositions concerning $\mathcal{W}(x)$, the main properties satisfied by $\precsim_L^\infty$ will then derive immediately.

Proposition 2. *$\mathcal{W}(x)$ can be expressed as a linear combination of the components of $L(x)$ using only strictly positive coefficients. We have:*
$\mathcal{W}(x) = \sum_{k=1}^{n-1}(w_k - w_{k+1})L_k(x) + w_n L_n(x) = w'.L(x)$
with $w' = (w_1 - w_2, w_2 - w_3, \ldots, w_{n-1} - w_n, w_n)$.

PROOF. Remarking that $x_{(1)} = L_1(x)$ and $x_{(k)} = L_k(x) - L_{k-1}(x)$ for $k = 2, ..., n.$, it is sufficient to make the substitution to get the desired linear combination with weights $w'_k = w_k - w_{k+1} > 0$ for $k < n$ and $w'_n = w_n > 0$. □

Proposition 3. *$\mathcal{W}$ is a Schur convex function.*

PROOF. We have to prove that $x \precsim_L y \Rightarrow \mathcal{W}(x) \leq \mathcal{W}(y)$. If $x \precsim_L y$ then by definition we have $L(x) \precsim_P L(y)$. Hence, considering the positive weighting vector w' used in the proof of Proposition 2, we have $w'_k.L_k(x) \leq w'_k.L_k(y)$ for $k = 1, ..., n$. After summing these n equalities, we get the result using proposition 2. □

This shows that $\precsim_L^\infty$ is based on a Schur convex function, like S-Gini indices and Yaari's model introduced in Example 3. As recalled before, Schur convex functions are known as convenient tools to measure inequalities in majorization theory (see [10]). We present now five properties satisfied by $\precsim_L^\infty$ which are consequences of Proposition 3. Property P1 shows that all vectors having the same Lorenz vector are treated equivalently:

P1: Neutrality. For all x, y in X, $L(x) = L(y) \Rightarrow x \sim_L^\infty y$.

Property P2 makes explicit the fact that $\prec_L^\infty$ is a refinement of Lorenz-dominance.

P2: Strict L-Monotonicity. $x \prec_L y \Rightarrow x \prec_L^\infty y$.

Then we introduce 3 axioms that better explain how $\precsim_L^\infty$ works with Lorenz vectors.

P3: Complete weak-order. $\precsim_L^\infty$ is reflexive, transitive and complete.

P4 Continuity. Let x, y, z be 3 cost-vectors such that $x \prec_L^\infty y \prec_L^\infty z$. There exists $\alpha, \beta \in]0,1[$ such that:

$$\alpha x + (1-\alpha)z \prec_L^\infty y \prec_L^\infty \beta x + (1-\beta)z.$$

PROOF. We have indeed $x \prec_L^\infty y \prec_L^\infty z \Rightarrow \mathcal{W}(x) < \mathcal{W}(y) < \mathcal{W}(z)$. Whenever $\alpha \to 1$ then the sequence of vectors of general term $\alpha x + (1-\alpha)z$ tends to x and, by continuity of $\mathcal{W}$, $\mathcal{W}(\alpha x + (1-\alpha)z) \to \mathcal{W}(x)$. Hence for α sufficiently close to 1, $\mathcal{W}(\alpha x + (1-\alpha)z)$ is sufficiently close to $\mathcal{W}(x)$ to be inferior to $\mathcal{W}(y)$. Hence $\alpha x + (1-\alpha)z \prec_L^\infty y$. We deliberately omit the other part of the proof that works similarly with $\beta \to 0$. □

The last property is a restriction to comonotonic vectors of the so-called independence axiom proposed by Von Neumann and Morgenstern [17] in the framework of utility theory. Comonotonicity of vectors is defined as follows:

Definition 6. *Two cost vectors x and y are said to be* comonotonic *if $x_i > x_j$ and $y_i < y_j$ for no $i, j \in \{1, \ldots, n\}$.*

Two solutions having comonotonic cost vectors satisfy the agents in the same order. It is useful to remark that, for any pair (x, y) of comonotonic vectors, there exists a permutation π of $\{1, \ldots, m\}$ such that $x_{\pi(1)} \geq x_{\pi(2)} \geq \ldots \geq x_{\pi(m)}$ and $y_{\pi(1)} \geq y_{\pi(2)} \geq \ldots \geq y_{\pi(m)}$. Consequently, $\mathcal{W}(\alpha x + (1-\alpha)y) = \alpha\mathcal{W}(x) + (1-\alpha)\mathcal{W}(y)$. We can now establish our last property:

P5 Comonotonic Independence. Let x, y, z 3 comonotonic cost vectors. Then, for all $\alpha \in]0,1[$:

$$x \prec_L^\infty y \Longrightarrow \alpha x + (1-\alpha)z \prec_L^\infty \alpha y + (1-\alpha)z.$$

PROOF. We have $x \prec_L^\infty y \Rightarrow \mathcal{W}(x) < \mathcal{W}(y)$. Hence $\mathcal{W}(\alpha x) < \mathcal{W}(\alpha y)$ and $\mathcal{W}(\alpha x) + \mathcal{W}((1-\alpha)z) < \mathcal{W}(\alpha y) + \mathcal{W}((1-\alpha)z)$. Since x and z are comonotonic we have $\mathcal{W}(\alpha x) + \mathcal{W}((1-\alpha)z) = \mathcal{W}(\alpha x + (1-\alpha)z)$. Moreover, y and z are comonotonic; hence we have $\mathcal{W}(\alpha y) + \mathcal{W}((1-\alpha)z) = \mathcal{W}(\alpha y + (1-\alpha)z)$. Finally we get $\mathcal{W}(\alpha x + (1-\alpha)z) < \mathcal{W}(\alpha y + (1-\alpha)z)$ and therefore $\alpha x + (1-\alpha)z \prec_L^\infty \alpha y + (1-\alpha)z$. □

Note that the restriction to comonotonic vectors is necessary within an independence axiom used for the measurement of inequalities [18]. If we forget it in the premisses of P5, we obtain a property which is incompatible with the Strict L-monotonicity axiom, as shown by the following:

Example 4. *Let us consider $x = (24, 24)$, $y = (22, 26)$ and $z = (26, 22)$ which are not comonotonic. Due to Strict L-monotonicity $x \prec_L^\infty y$. Hence, usual independence would imply $(25, 23) = \frac{1}{2}x + \frac{1}{2}z \prec_L^\infty \frac{1}{2}y + \frac{1}{2}z = (24, 24)$ which is in contradiction with $(24, 24) \prec_L (25, 23)$.*

The above properties exhibit nice features of L^∞-dominance and underline some relationships with Yaari's model introduced in Example 3. This is natural because all these models are based on OWA operators with decreasing weights.

We have shown that fair optimization in multiagent problems can reasonably be formulated as minimizing function $\mathcal{W}(x)$ over feasible cost vectors. However $\mathcal{W}(x)$ is not a linear function since, for non-comonotonic vectors x, y, $\mathcal{W}(x + y) \neq \mathcal{W}(x) + \mathcal{W}(y)$ in general. Hence minimizing $\mathcal{W}(x)$ requires non-linear optimization. The next section is devoted to this point in the context of many to many multiagent assignment problems.

4. SOLVING MULTIAGENT ASSIGMENT PROBLEMS

The general many to many multiagent assignment problem we are considering can be stated as follows: we want to assign m items to n agents. The number of items assigned to agent i is restricted to interval $[l_i, u_i]$, $i = 1, \ldots, n$. Item j must be assigned to a number of agents restricted to the interval $[l'_j, u'_j]$, $j = 1, \ldots, m$. A $n \times m$ matrix gives the cost c_{ij} of assigning item j to agent i.

This general problem occurs in many contexts such as paper assignment problems, social meeting on the web, resource allocation, transportation problems. The possible solutions are characterized by a $n \times m$ matrix of booleans z_{ij} representing the possibility of assigning item j to individual i. Hence the problem can be formalized as a multiobjective 0-1 linear optimization problem:

$$\text{Min} \;\; x_i = \sum_{j=1}^{m} c_{ij} z_{ij}, \quad i = 1, \ldots, n$$

$$s.t. \begin{cases} l'_j \leq \sum_{i=1}^{n} z_{ij} \leq u'_j & j = 1, \ldots, m \\ l_i \leq \sum_{j=1}^{m} z_{ij} \leq u_i & i = 1, \ldots, n \\ z_{ij} \in \{0, 1\} & \forall i, \forall j \end{cases}$$

This general multiobjective programm fits to many different situations involving multiple agents. For example, in fair allocation of indivisible goods, we set $l'_j = u'_j = 1$, $j = 1, \ldots, m$. This is the case when we have to distribute presents to kids as in the Santa Claus problem [1], or in some auctions problems. Alternatively, the multiobjective problem can easily model a conference paper assignment problem. In this case $l'_j = u'_j = 3$, $j = 1, \ldots, m$ (a paper must be reviewed 3 PC members) and u_i (resp. l_i) represent the maximal (resp. minimal) number of papers we want to assign to reviewer i. In this case, x_i represents the overall charge or dissatisfaction of agent i.

In such multiagent combinatorial problems, the number of Pareto-optimal solutions and L-non-dominated solutions can be large, as illustrated in the following example.

Example 5. *Consider a particular instance of the above problem with m items to be assigned to 2 agents ($n = 2$). Assume that $l_1 = l_2 = 0$, $u_1 = u_2 = n$, $l'_j = u'_j = 1$, $j = 1, \ldots, m$ with costs $c_{1j} = 2^j$ and $c_{2j} = 2^{j-1}$, $j = 1 \ldots m-1$, $c_{1m} = 4^m$, $c_{2m} = 2^m + 1$. This gives 2^m distinct feasible assignments. The half of them assigns item m to agent 1 which is prohibitive. All of them are L-dominated. The other half produces cost vectors $\{(2k, 3 \times 2^{m-1} - k), k \in \{0, \ldots, 2^{m-1} - 1\}$. Note that $3 \times 2^{m-1} - k > 2k$. Consequently, the associate Lorenz vectors are $\{(3 \times 2^{m-1} - k, 3 \times 2^{m-1} + k), k \in \{0, \ldots, 2^{m-1} - 1\}\}$. No P-dominance holds between these Lorenz vectors because their sum of components is constant. Hence we have 2^{m-1} L-non-dominated solutions.*

In this family of instances, the number of L-non-dominated feasible cost vectors grows exponentially with m. Even if we want only one feasible solution for each distinct cost vector, the size of the output set remains exponential in m. Clearly, L^∞-dominance can help to reduce the set of optimal solutions.

4.1 Fair Multiagent Optimization

Using the result established in Theorem 2, the search of an optimal many to many assignment problem with respect to L^∞-dominance can be formulated as the following 0-1 non-linear optimization problem (Π):

$$\text{Min} \;\; \mathcal{W}(x) = \sum_{k=1}^{n} \sin\left(\frac{(n+1-k)\pi}{2n+1}\right) x_{(k)} \quad (2)$$

$$(\Pi) \qquad s.t. \begin{cases} x_i = \sum_{j=1}^{m} c_{ij} z_{ij} & i = 1, \ldots, n \\ l'_j \leq \sum_{i=1}^{n} z_{ij} \leq u'_j & j = 1, \ldots, m \\ l_i \leq \sum_{j=1}^{m} z_{ij} \leq u_i & i = 1, \ldots, n \\ z_{ij} \in \{0, 1\} & \forall i, \forall j \end{cases} \quad (3)$$

Proposition 4. *The problem P_α consisting in deciding whether there exists an assignment with cost $\mathcal{W}(x) \leq \alpha$ is an NP-complete decision problem for any fixed positive α.*

Proof. P_α is clearly in NP. To establish NP-completeness, we reduce the NP-complete Partition Problem to our problem. The Partition Problem is stated as follows:

Instance: finite set $A = \{a_1, \ldots, a_m\}$ of items and a size $s(a) \in \mathbb{N}$ for each $a \in A$.
Question: is it possible to partition A into two sets of objects of equal weights?
From an instance of Partition Problem, we construct in polynomial time an instance of P_α with $n = 2$, $l_1 = l_2 = 0$, $u_1 = u_2 = m$, $l'_j = u'_j = 1$, and $c_{1j} = c_{2j} = s(a_j)$, $j = 1, \ldots, m$. Moreover we set $\alpha = (w_1 + w_2)\beta$ with $\beta = \sum_{a \in A} s(a)/2$. Hence, the answer to P_α is YES if and only if the answer to the partition problem is YES. Indeed, if there is a solution to the partition problem, then there exists an assignment with cost (β, β) and $\mathcal{W}(\beta, \beta) = \alpha$. Moreover, if the answer to the partition problem is NO, then any partition of A into two subsets is unfair and the corresponding assignment leads to a cost vector of type $(\beta - \varepsilon, \beta + \varepsilon)$, $\varepsilon \in (0, \beta]$. Since $(\beta, \beta) \prec_L (\beta - \varepsilon, \beta + \varepsilon)$ we have $\mathcal{W}(\beta - \varepsilon, \beta + \varepsilon) > \mathcal{W}(\beta, \beta) = \alpha$. So there is no assignment such that $\mathcal{W}(x) = \alpha$; the answer to P_α is NO. □

4.2 Linearization of the problem

Thanks to Proposition 2, Π rewrites:

$$(\Pi') \qquad \text{Min} \; \mathcal{W}(x) = \sum_{k=1}^{n} w'_k L_k(x) \;\; s.t. \;\; (3)$$

with $w' = (w_1 - w_2, w_2 - w_3, \ldots, w_{n-1} - w_n, w_n)$. Following an idea introduced in [13], we express the k^{th} component $L_k(x)$ of the Lorenz vector $L(x)$ as the solution of the following linear program:

$$\text{Max} \;\; \left(\sum_{i=1}^{n} \alpha_{ik} x_i\right) \quad s.t. \begin{cases} \sum_{i=1}^{n} \alpha_{ik} = k \\ 0 \leq \alpha_{ik} \leq 1 & i = 1 \ldots n \end{cases}$$

Its optimal value is clearly the sum of the k greatest components of x, that is $L_k(x)$. This is also the optimal value of the dual problem:

$$\text{Min} \;\; \left(k \; r_k + \sum_{i=1}^{n} b_{ik}\right) \quad s.t. \begin{cases} r_k + b_{ik} \geq x_i & i = 1 \ldots n \\ b_{ik} \geq 0 & i = 1 \ldots n \end{cases}$$

We can therefore combine the linear program above with Π' (since both are in minimization and $w' > 0$) and rewrite our problem Π as the following mixed integer linear program:

$$\text{Min} \;\; \sum_{k=1}^{n} w'_k \left(k \times r_k + \sum_{i=1}^{n} b_{ik}\right)$$

$$(\Gamma) \qquad s.t. \begin{cases} l'_j \leq \sum_{i=1}^{n} z_{ij} \leq u'_j & j = 1, \ldots, m \\ l_i \leq \sum_{j=1}^{m} z_{ij} \leq u_i & i = 1, \ldots, n \\ r_k + b_{ik} \geq \sum_{j=1}^{m} c_{ij} z_{ij} & \forall i, k = 1, \ldots, n \\ b_{ik} \geq 0 & \forall i, \forall k \\ z_{ij} \in \{0, 1\} & \forall i, \forall j \end{cases}$$

Γ has $2(n^2 + m + n)$ constraints, nm 0-1 variables, and $n^2 + n$ continuous variables.

4.3 Numerical Tests

We present here numerical tests[2] performed on random instances of one-to-one and many-to-many multiagent assignment problems. To solve the mixed integer linear program Γ, we used ILOG CPLEX 11.100 on a computer with

[2]The authors wish to thank Julien Lesca (LIP6-UPMC) for his participation to numerical tests.

4 Go of memory and an Intel Core 2 Duo 2.66 GHz processor. Table 1 (resp. Table 2) gives the results obtained for the assignment of m objects to $n = m$ agents, with $l_i = l'_i = u_i = u'_i = 1$ and costs randomly generated in $[1, 1000]$ (resp. $[1, 20]$). Table 3 is the test on the paper assignment problem modeled as follows: $n = m/4$, each reviewer receives at most 9 papers ($l_i = 0$ and $u_i = 9$), a paper has to be reviewed by exactly 2 reviewers ($l'_j = u'_j = 2$), and a reviewer expresses his preferences for reviewing a paper with a number between 0 and 5 (i.e. costs are in $[0, 5]$). The computation times expressed in seconds represent average solution times over 20 random instances of the same size m (number of objects) with a timeout set to 1000 seconds.

m	t
10	.01
20	.09
30	.33
40	1.52
50	5.14
60	16.1
70	34.0
80	81.8
90	136
100	275

1. Costs in $[1, 1000]$

m	t
100	.93
200	3.65
300	17.4
400	52.8
500	104
600	161
700	390
800	482
900	843
1000	>1000

2. Costs in $[1, 20]$

m	t
200	3.51
300	5.63
400	13.9
500	35.7
600	79.4
700	148
800	303
900	478
1000	904
1100	>1000

3. Paper Assignment

So, it is possible to find a fair solution to the paper assignment problem with realistic parameters for a standard conference within a reasonable time. The approach presented here remains valid for finding fair assignments by optimization of S-Gini indices and other instances of the Yaari's model. Indeed, as ordered weighted averages, such indices can be linearized similarly as $\mathcal{W}$. We have performed tests showing that solution times using such criteria are in the same order of magnitude.

5. CONCLUSION

We have investigated the notion of infinite order Lorenz dominance and its use in fair multiagent assignment problems. The representation result established in Section 3 makes it possible to formulate the search of non-dominated solutions as a single-objective optimization problem. We have used this result to solve assignment problems and shown the effectiveness of the approach on non-trivial combinatorial problems involving a significant number agents.

The representation of L^∞-dominance by an OWA operator with strictly positive and decreasing weights is easily interpretable: $\mathcal{W}$ is an intermediate between the Max operator generally used to model egalitarism and weighted averages used to model utilitarism. Although the former focuses on the least satisfied agent, the latter is fully compensatory and does not provide any control on fairness in the distribution of costs. Thus, $\mathcal{W}$ provides a soft compromise between these two extreme attitudes, putting more weight on least satisfied agents while keeping some compensation possibilities.

Our results can be extended to the case of weighted agents (which occurs, for example, in resource allocation problems, where agents can have exogenous rights represented by individual weights). It is possible to show that the weighted extension of L-dominance converges by iteration towards a weighted extension of OWA. The associate optimization problem can be solved efficiently by slightly modifying the mixed integer linear program Γ.

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